

A new approach to Poincaré-type inequalities on the Wiener space

ALBERTO LANCONELLI

DIPARTIMENTO DI MATEMATICA
UNIVERSITA' DEGLI STUDI DI BARI
VIA E. ORABONA, 4
70125 BARI - ITALY
E-MAIL: *alberto.lanconelli@uniba.it*

Abstract

We prove a new type of Poincaré inequality on abstract Wiener spaces for a family of probability measures which are absolutely continuous with respect to the reference Gaussian measure. This class of probability measures is characterized by the strong positivity (a notion introduced by Nualart and Zakai in [17]) of their Radon-Nikodym densities. Measures of this type do not belong in general to the class of log-concave measures, which are a wide class of measures satisfying the Poincaré inequality (Brascamp and Lieb [2]). Our approach is based on a point-wise identity relating Wick and ordinary products and on the notion of strong positivity which is connected to the non negativity of Wick powers. Our technique leads also to a partial generalization of the Houdré and Kagan [8] and Houdré and Pérez-Abreu [9] Poincaré-type inequalities.

Keywords: Poincaré inequality, abstract Wiener space, Wick product, stochastic exponentials.

Mathematics Subject Classification (2000): 60H07, 60H30, 60H40.

1 Introduction

The *Poincaré* or *spectral gap* inequality for the standard n -dimensional Gaussian measure states that

$$\int_{\mathbb{R}^n} f^2(x) d\mu(x) - \left(\int_{\mathbb{R}^n} f(x) d\mu(x) \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 d\mu(x) \quad (1.1)$$

where f is a smooth function, $d\mu(x) = (2\pi)^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{2}\} dx$ and ∇ denotes the gradient operator. Inequality (1.1) was first proved by Nash in [15] and later on rediscovered by Chernoff in [3].

The literature concerning extensions, refinements and applications of the Poincaré inequality is very rich. One of the key features of inequality (1.1) is that it is dimension

independent and in fact one can prove (see Gross [7] also for the connection with logarithmic Sobolev inequalities) its validity on abstract Wiener spaces (see Section 2 below). An important refinement to inequality (1.1) was proposed by Houdré and Kagan in [8] where they proved the following inequalities

$$\begin{aligned} \sum_{l=1}^{2k} \frac{(-1)^{l+1}}{l!} \int_{\mathbb{R}^n} |\nabla^l f(x)|^2 d\mu(x) &\leq \int_{\mathbb{R}^n} f^2(x) d\mu(x) - \left(\int_{\mathbb{R}^n} f(x) d\mu(x) \right)^2 \\ &\leq \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} \int_{\mathbb{R}^n} |\nabla^l f(x)|^2 d\mu(x). \end{aligned} \quad (1.2)$$

Here ∇^l stands for iterated gradients and $|\cdot|$ are the Euclidean norms on the corresponding \mathbb{R}^n spaces. Later on this result was extended to the classical Wiener space by Houdré and Pérez-Abreu in [9].

One of the most celebrated generalizations of the Poincaré inequality (1.1) is due to Brascamp and Lieb [2] which extended the validity of (1.1) to the class of log-concave measures. More precisely they proved that if ν is a probability measure on \mathbb{R}^n of the form $d\nu(x) = e^{-V(x)} dx$, for some smooth strictly convex function V , then

$$\int_{\mathbb{R}^n} f^2(x) d\nu(x) - \left(\int_{\mathbb{R}^n} f(x) d\nu(x) \right)^2 \leq \int_{\mathbb{R}^n} \langle (\mathcal{H}V(x))^{-1} \nabla f(x), \nabla f(x) \rangle d\nu(x), \quad (1.3)$$

where $\mathcal{H}V$ is the Hessian matrix of V . The previous inequality was adapted to the context of abstract Wiener spaces by Feyel and Üstünel in [6].

The aim of the present paper is to prove a new type of Poincaré inequality for a class of probability measures on an abstract Wiener space which is not included in the family of log-concave measures. We are concerned with probability measures that are absolutely continuous with respect to the reference Gaussian measure and with Radon-Nikodym densities that satisfy a strong positivity requirement (see Section 3 below). This strong positivity condition is the crucial ingredient of our approach; the notion was introduced in Nualart and Zakai [17] and it is connected to the non negativity of Wick powers (see Theorem 3.4 below). It is in fact a formula involving the Wick product to be the source of inspiration for this paper; the formula in its simplest appearance reads

$$(f \diamond f)(x) = f(x)^2 + \sum_{l \geq 1} \frac{(-1)^l}{l!} [f^{(l)}(x)]^2. \quad (1.4)$$

(Here $f^{(l)}$ denotes the l -th derivative of f). Identity (1.4) appeared for the first time in Nualart and Zakai [17] and independently in Hu and Øksendal [11], both times without proof; see Hu and Yan [12] and Lanconelli [14] for proofs in different settings.

Taking the integral of both sides of (1.4) with respect to the one dimensional Gaussian measure and using the identity

$$\int_{\mathbb{R}} (f \diamond f)(x) d\mu(x) = \left(\int_{\mathbb{R}} f(x) d\mu(x) \right)^2$$

one gets

$$\left(\int_{\mathbb{R}} f(x) d\mu(x) \right)^2 = \int_{\mathbb{R}} f(x)^2 d\mu(x) + \sum_{l \geq 1} \frac{(-1)^l}{l!} \int_{\mathbb{R}} [f^{(l)}(x)]^2 d\mu(x),$$

which coincides with (1.2) if one formally let k going to infinity.

The paper is organized as follows: In Section 2 we introduce the notation and background material needed to develop our approach to the Poincaré inequality ; Section 3 is devoted to the key notion of strong positivity with several illustrating examples; finally in Section 4 we state and prove the main result of the paper (Theorem 4.1) together with a refinement (Theorem 4.5) in the spirit of inequality (1.2)

2 Framework

The aim of this section is to collect the necessary background material and fix the notation. For the sake of clarity the topics will not be treated in their greatest generality but they will be developed in relation to the application we have in mind, the Poincaré inequality. For more details the interested reader is referred to the books of Nualart [16], Janson [13] and to the paper by Potthoff and Timpel [18] (the latter reference is suggested, among other things, for the theory of the spaces \mathcal{G}_λ and the notion of Wick product).

2.1 The spaces $\mathbb{D}^{k,p}$ and \mathcal{G}_λ

Let (H, W, μ) be an *abstract Wiener space*, that means $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space which is continuously and densely embedded in the Banach space $(W, \|\cdot\|_W)$ and μ is a Gaussian probability measure on the Borel sets $\mathcal{B}(W)$ of W such that

$$\int_W e^{i\langle w, w^* \rangle} d\mu(w) = e^{-\frac{1}{2}\|w^*\|_H^2}, \quad \text{for all } w^* \in W^*. \quad (2.1)$$

Here $W^* \subset H$ denotes the dual space of W (which in turn is dense in H) and $\langle \cdot, \cdot \rangle$ stands for the dual pairing between W and W^* . We will refer to H as the *Cameron-Martin* space of W . Set for $p \geq 1$

$$\mathcal{L}^p(W, \mu) := \left\{ F : W \rightarrow \mathbb{R} \text{ such that } \|F\|_p := \left(\int_W |F(w)|^p d\mu(w) \right)^{\frac{1}{p}} < +\infty \right\}.$$

To ease the notation we will write $E[F]$ for $\int_W F(w) d\mu(w)$ and call it the *expectation* of F . It follows from (2.1) that the map

$$\begin{aligned} W^* &\rightarrow \mathcal{L}^2(W, \mu) \\ w^* &\mapsto \langle w, w^* \rangle \end{aligned}$$

is an isometry; we can therefore define for μ -almost all $w \in W$ the quantity $\langle w, h \rangle$ for $h \in H$ as an element of $\mathcal{L}^2(W, \mu)$. This element will be denoted by $\delta(h)$.

We now introduce the gradient operator and the class of functions for which our Poincaré inequality will hold true. On the set

$$\mathcal{S} := \{ F = \varphi(\delta(h_1), \dots, \delta(h_n)) \text{ where } n \in \mathbb{N}, h_1, \dots, h_n \in H \text{ and } \varphi \in C_0^\infty(\mathbb{R}^n) \}$$

define

$$D(\varphi(\delta(h_1), \dots, \delta(h_n))) := \sum_{k=0}^n \frac{\partial \varphi}{\partial x_j}(\delta(h_1), \dots, \delta(h_n)) h_j.$$

The operator D maps \mathcal{S} into $\mathcal{L}^p(W, \mu; H)$; moreover by means of the integration by parts formula

$$E[\langle DF, h \rangle_H] = E[F\delta(h)], \quad F \in \mathcal{S}, h \in H$$

one can prove that D is closable in $\mathcal{L}^p(W, \mu)$; we therefore define the space $\mathbb{D}^{1,p}$ to be closure of \mathcal{S} under the norm

$$\|F\|_{1,p} := \left(E[|F|^p] + E[\|DF\|_H^p] \right)^{\frac{1}{p}}.$$

In a similar way, iterating the definition of D and introducing for any $k \in \mathbb{N}$ the norms

$$\|F\|_{k,p} := \left(E[|F|^p] + \sum_{j=1}^k E[\|D^j F\|_{H^{\otimes j}}^p] \right)^{\frac{1}{p}}.$$

one constructs the spaces $\mathbb{D}^{k,p}$.

In order to prove our main results we need to introduce an additional class of functions. This will be related to the family of probability measures with respect to which the Poincaré inequality will be proved. To this aim recall that by the Wiener-Itô chaos decomposition theorem any element F in $\mathcal{L}^2(W, \mu)$ has an infinite orthogonal expansion

$$F = \sum_{n \geq 0} \delta^n(f_n),$$

where $f_n \in H^{\hat{\otimes} n}$, the space of symmetric elements of $H^{\otimes n}$, and $\delta^n(f_n)$ stands for the multiple Wiener-Itô integral of f_n . We remark that $\delta^1(f_1)$ coincides with the element $\delta(f_1)$ mentioned above. Moreover one has

$$\|F\|_2^2 = \sum_{n \geq 0} n! \|f_n\|_{H^{\otimes n}}^2.$$

It is useful to observe that if F happens to be in $\mathbb{D}^{1,2}$ then

$$E[\|DF\|_H^2] = \sum_{n \geq 1} n n! \|f_n\|_{H^{\otimes n}}^2.$$

For any $\lambda \geq 0$ define the operator $\Gamma(\lambda)$ acting on $\mathcal{L}^2(W, \mu)$ as

$$\Gamma(\lambda) \left(\sum_{n \geq 0} \delta^n(f_n) \right) := \sum_{n \geq 0} \lambda^n \delta^n(f_n).$$

If $\lambda \leq 1$ then $\Gamma(\lambda)$ coincides with the Ornstein-Uhlenbeck semigroup

$$(P_t f)(w) := \int_W f(e^{-t}w + \sqrt{1 - e^{-2t}}\tilde{w}) d\mu(\tilde{w}), \quad w \in W, t \geq 0$$

(take $\lambda = e^{-t}$) which is a bounded operator. Otherwise $\Gamma(\lambda)$ is an unbounded operator with domain

$$\mathcal{G}_\lambda := \left\{ F = \sum_{n \geq 0} \delta^n(f_n) \in \mathcal{L}^2(W, \mu) \text{ such that } \|F\|_{\mathcal{G}_\lambda}^2 := \sum_{n \geq 0} n! \lambda^{2n} \|f_n\|_{H^{\otimes n}}^2 < +\infty \right\}.$$

The family $\{\mathcal{G}_\lambda\}_{\lambda \geq 1}$ is a collection of Hilbert spaces with the property that

$$\mathcal{G}_{\lambda_2} \subset \mathcal{G}_{\lambda_1} \subset \mathcal{L}^2(W, \mu)$$

for $1 < \lambda_1 < \lambda_2$. Define $\mathcal{G} := \bigcap_{\lambda \geq 1} \mathcal{G}_\lambda$ endowed with the projective limit topology; the space \mathcal{G} turns out to be a reflexive Fréchet space. Its dual \mathcal{G}^* is a space of generalized functions that can be represented as $\mathcal{G}^* = \bigcup_{\lambda > 0} \mathcal{G}_\lambda$. We remark that for $F \in \mathcal{L}^2(W, \mu)$ and $G \in \mathcal{G}$ one has

$$\langle\langle F, G \rangle\rangle = E[F \cdot G].$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ stands for the dual pairing between \mathcal{G}^* and \mathcal{G} . It also follows from our construction that for $1 < \lambda_1 < \lambda_2$,

$$\mathcal{G} \subset \mathcal{G}_{\lambda_2} \subset \mathcal{G}_{\lambda_1} \subset \mathcal{L}^2(W, \mu) \subset \mathcal{G}_{\frac{1}{\lambda_1}} \subset \mathcal{G}_{\frac{1}{\lambda_2}} \subset \mathcal{G}^*.$$

Here $\mathcal{G}_{\frac{1}{\lambda_i}}$ represents the dual space of \mathcal{G}_{λ_i} , $i = 1, 2$, respectively.

Remark 2.1 *It is not difficult to see that for $\lambda > 1$ we have $\mathcal{G}_\lambda \subset \bigcap_{k \geq 1} \mathbb{D}^{k,2}$. Now, in the context of the classical Wiener space (i.e. H is the space of absolutely continuous functions on $[0, 1]$ with square integrable derivative and which are zero at zero and W is the space of continuous functions on $[0, 1]$ which are also zero at zero) the Stroock formula (see [19]) tells that if $F \in \bigcap_{k \geq 1} \mathbb{D}^{k,2}$ then the elements f_n appearing in its chaos decomposition are explicitly given by*

$$f_n(t_1, \dots, t_n) = \frac{1}{n!} \int_0^{t_1} \cdots \int_0^{t_n} E[D_{s_1, \dots, s_n} F] ds_1 \cdots ds_n$$

where $D_{s_1, \dots, s_n} F$ stands for the n -th Malliavin derivative of F . Using this representation together with the Jensen inequality we get a sufficient condition for F to be in \mathcal{G}_λ :

Let $F \in \bigcap_{k \geq 1} \mathbb{D}^{k,2}$ be such that the series

$$\sum_{n \geq 0} \frac{\lambda^{2n}}{n!} E[\|D^n F\|_{H^{\otimes n}}^2]$$

converges. Then $F \in \mathcal{G}_\lambda$.

One of the most representative elements of \mathcal{G} is the so called *stochastic exponential*

$$\mathcal{E}(h) := \exp \left\{ \delta(h) - \frac{1}{2} \|h\|_H^2 \right\}, \quad h \in H.$$

(We recall that stochastic exponentials correspond among other things to Radon-Nikodym derivatives, with respect to the underlying Gaussian measure μ , of probability measures on $(W, \mathcal{B}(W))$ obtained through shifted copies of μ along Cameron-Martin directions). Its membership to \mathcal{G} can be easily verified since the Wiener-Itô chaos decomposition of $\mathcal{E}(h)$ is obtained with $f_n = \frac{h^{\otimes n}}{n!}$.

Moreover the linear span of the stochastic exponentials, that we denote with \mathcal{E} , is dense in $\mathcal{L}^p(W, \mu)$, $\mathbb{D}^{k,p}$, for any $p \geq 1$ and $k \in \mathbb{N}$, and \mathcal{G} .

2.2 The Wick product

For $h, k \in H$ define

$$\mathcal{E}(h) \diamond \mathcal{E}(k) := \mathcal{E}(h + k).$$

This is called the *Wick product* of $\mathcal{E}(h)$ and $\mathcal{E}(k)$. Extend this operation by linearity to \mathcal{E} , the linear span of stochastic exponentials, to get a commutative, associative and distributive (with respect to the sum) multiplication.

Now let $F, G \in \mathcal{L}^2(W, \mu)$ and $F_n, G_n \in \mathcal{E}$ such that

$$\lim_{n \rightarrow +\infty} \|F_n - F\|_2 = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|G_n - G\|_2 = 0.$$

Set

$$F \diamond G := \lim_{n \rightarrow +\infty} F_n \diamond G_n.$$

The limit above can not be interpreted in the $\mathcal{L}^2(W, \mu)$ -norm since the Wick product is easily seen to be an unbounded bilinear operator on that space. The Wick product $F \diamond G$ of the two square integrable elements F and G lives in the distributional space \mathcal{G}^* ; more precisely,

$$F, G \in \mathcal{L}^2(W, \mu) \quad \Rightarrow \quad F \diamond G \in \mathcal{G}_{\frac{1}{\sqrt{2}}}. \quad (2.2)$$

One can also prove that the Wick product is a continuous bilinear operator on $\mathcal{G} \times \mathcal{G}$ and on $\mathcal{G}^* \times \mathcal{G}^*$.

We mention that for any $F \in \mathbb{D}^{1,2}$ and $h \in H$ one has $F \diamond \delta(h) \in \mathcal{L}^2(W, \mu)$; moreover the following identity holds

$$\begin{aligned} F \diamond \delta(h) &= F \cdot \delta(h) - \langle DF, h \rangle_H \\ &= D_h^* F \end{aligned}$$

where D_h^* is the formal adjoint of $\langle D\cdot, h \rangle_H$ in $\mathcal{L}^2(W, \mu)$.

A characterizing property of the Wick product is the following: for any $F, G \in \mathcal{L}^2(W, \mu)$ and $h \in H$,

$$\begin{aligned} \langle \langle F \diamond G, \mathcal{E}(h) \rangle \rangle &= \langle \langle F, \mathcal{E}(h) \rangle \rangle \cdot \langle \langle G, \mathcal{E}(h) \rangle \rangle \\ &= E[F\mathcal{E}(h)] \cdot E[G\mathcal{E}(h)] \end{aligned} \quad (2.3)$$

In particular for $h = 0$ one gets

$$\langle \langle F \diamond G, 1 \rangle \rangle = E[F] \cdot E[G]. \quad (2.4)$$

We refer the reader to the papers of Da Pelo et al. [4] and [5] for other properties and deeper results on the Wick product.

3 Strongly positive functions

We now introduce a concept that will play a crucial role in our approach to the Poincaré inequality.

Definition 3.1 A generalized function $\Phi \in \mathcal{G}^*$ is said to be positive if for any $\varphi \in \mathcal{G}$ one has $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$.

A generalized function $\Phi \in \mathcal{G}^*$ is said to be strongly positive if for any $\lambda \geq 1$, the generalized function $\Gamma(\lambda)\Phi$ is positive.

The notion of strong positivity was introduced by Nualart and Zakai in [17] and it is related to the positivity improving property of the Ornstein-Uhlenbeck semigroup. Observe that if F is strongly positive then it is also positive (according to Definition 3.1). Moreover $F \in \mathcal{L}^p(W, \mu) \subset \mathcal{G}^*$ for some $p > 1$ is positive according to Definition 3.1 if and only if F is non negative in the usual sense, i.e. $\mu(F \geq 0) = 1$.

Example 3.2 Any element of the form $\mathcal{E}(h)$, $h \in H$ is strongly positive; in fact

$$\Gamma(\lambda)\mathcal{E}(h) = \mathcal{E}(\lambda h)$$

and therefore $\Gamma(\lambda)\mathcal{E}(h)$ is non negative with probability one for all $\lambda \geq 1$. In particular, taking $h = 0$ we get that non negative constants are strongly positive.

It is also straightforward to note that convex combinations of stochastic exponentials are strongly positive (this is actually a particular case of Theorem 5.1 in [17]).

We now consider a more interesting example

Example 3.3 Let $\{W_{t,x}\}_{0 \leq t \leq T, x \in \mathbb{R}}$ be a space-time white noise and consider the following stochastic partial differential equation

$$\begin{aligned} \partial_t u(t, x) &= \frac{1}{2} \partial_{xx} u(t, x) + u(t, x) \diamond W_{tx}, \quad t \in]0, T], x \in \mathbb{R} \\ u(0, x) &= 1, \quad x \in \mathbb{R}. \end{aligned}$$

Then the unique square integrable mild solution to the previous Cauchy problem is a strongly positive element of \mathcal{G} .

To see this, we proceed at a quite formal level since a rigorous proof of the previous statement is beyond the scope of this paper (but can be deduced easily from the results of Hu [10]).

For any $\lambda \in \mathbb{R}$ denote with $\{u^\lambda(t, x)\}_{0 \leq t \leq T, x \in \mathbb{R}}$ the unique square integrable mild solution of the Cauchy problem

$$\begin{aligned} \partial_t u^\lambda(t, x) &= \frac{1}{2} \partial_{xx} u^\lambda(t, x) + \lambda u^\lambda(t, x) \diamond W_{tx}, \quad t \in [0, T[, x \in \mathbb{R} \\ u^\lambda(0, x) &= 1, \quad x \in \mathbb{R}. \end{aligned}$$

It is easy to see that $u^\lambda(t, x) = \Gamma(\lambda)u(t, x)$ and that this last equality implies that for any $\lambda \in \mathbb{R}$, $\Gamma(\lambda)u(t, x) \in \mathcal{L}^2(W, \mu)$ which is equivalent to say that $u(t, x)$ belongs to \mathcal{G} .

Moreover, since it is known (see Bertini and Cancrini [1]) that for any $\lambda \in \mathbb{R}$ the random variable $u^\lambda(t, x)$ is strictly positive with probability one, from $u^\lambda(t, x) = \Gamma(\lambda)u(t, x)$ we also get the strong positivity of $u(t, x)$.

The next result is a slight modification of Proposition 5.1 in [17]. It becomes a characterization of strong positivity when we replace stochastic exponentials with their complex counterparts

Theorem 3.4 *If $\Phi \in \mathcal{G}^*$ is strongly positive then*

$$\langle\langle \Phi, \varphi \diamond \varphi \rangle\rangle \geq 0 \text{ for all } \varphi \in \mathcal{E}. \quad (3.1)$$

Since \mathcal{E} is dense in \mathcal{G} and the Wick product is a continuous bilinear operator on that space, condition (3.1) is equivalent to

$$\langle\langle \Phi, \varphi \diamond \varphi \rangle\rangle \geq 0 \text{ for all } \varphi \in \mathcal{G}.$$

In the sequel, strongly positive functions will play the role of Radon-Nikodym densities of probability measures on $(W, \mathcal{B}(W))$ with respect to the reference Gaussian measure μ . We now want to show that measures of this type do not belong in general to the class of log-concave measures.

To this aim consider the finite dimensional abstract Wiener space given by $H = W = \mathbb{R}^n$, $n \in \mathbb{N}$ and

$$\mu(A) := \int_A (2\pi)^{-\frac{n}{2}} e^{-\frac{|w|^2}{2}} dw, \quad A \in \mathcal{B}(\mathbb{R}^n), \quad (3.2)$$

where $|\cdot|$ denotes the n -th dimensional Euclidean norm.

In this framework the class of stochastic exponentials is represented by the functions

$$\mathcal{E}(x) := e^{\langle x, w \rangle - \frac{|x|^2}{2}}, \quad w \in \mathbb{R}^n$$

where x is arbitrarily chosen in \mathbb{R}^n and $\langle \cdot, \cdot \rangle$ denotes, only for this particular example, the scalar product on \mathbb{R}^n .

As we mentioned before in Example 3.2 convex combinations of stochastic exponentials are strongly positive. Therefore if we define

$$\varphi(w) = \frac{\mathcal{E}(a) + \mathcal{E}(b)}{2}$$

for some $a, b \in \mathbb{R}^n$ then φ is a strongly positive function. Now, look at φ as a Radon-Nikodym density of a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with respect to the Gaussian measure μ defined in (3.2); multiplying φ by the Gaussian density appearing in the integral (3.2) we will obtain the density of the above mentioned measure with respect to the n -dimensional Lebesgue measure, i.e.

$$\begin{aligned} \varphi(w) \cdot (2\pi)^{-\frac{n}{2}} e^{-\frac{|w|^2}{2}} &= \frac{\mathcal{E}(a) + \mathcal{E}(b)}{2} \cdot (2\pi)^{-\frac{n}{2}} e^{-\frac{|w|^2}{2}} \\ &= \frac{e^{\langle a, w \rangle - \frac{|a|^2}{2}} + e^{\langle b, w \rangle - \frac{|b|^2}{2}}}{2} \cdot (2\pi)^{-\frac{n}{2}} e^{-\frac{|w|^2}{2}} \\ &= (2\pi)^{-\frac{n}{2}} \frac{e^{-\frac{|w-a|^2}{2}} + e^{-\frac{|w-b|^2}{2}}}{2} \end{aligned} \quad (3.3)$$

If $a \neq b$ then the function appearing in the last term of the above chain of equality is not log-concave, i.e. it is not the exponential of a concave function.

This shows that probability measures with strongly positive densities with respect to the standard Gaussian measure do not belong in general to the class of log-concave measures.

4 Main results

We are now ready to state and prove the first main result of the present paper. In the sequel the symbol $E_\nu[\cdot]$ will denote the expectation with respect to the measure $\nu d\mu$.

Theorem 4.1 *Let $\nu \in \mathcal{G}_{\sqrt{2}}$ be strongly positive. Then for any $F \in \mathbb{D}^{1,3}$ we have*

$$0 \leq E_\nu[|F|^2] - \langle \langle F \diamond F, \nu \rangle \rangle \leq E_\nu[\|DF\|_H^2]. \quad (4.1)$$

Before proving the theorem above we would like to justify through a simple example the appearance of the quantity $\langle \langle F \diamond F, \nu \rangle \rangle$ in (4.1).

Consider the function in (3.3) which was shown to be an admissible candidate for our Poincaré inequality but not fitting in the framework of the Brascamp-Lieb inequality (1.3) since it not log-concave. The function in (3.3) is a convex combination of the two log-concave densities

$$p(w - a) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|w-a|^2}{2}} \quad \text{and} \quad p(w - b) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|w-b|^2}{2}},$$

where we set $p(w) := (2\pi)^{-\frac{n}{2}} e^{-\frac{|w-b|^2}{2}}$. Therefore the measures

$$d\nu_a(w) := p(w - a)dw \quad \text{and} \quad d\nu_b(w) := p(w - b)dw$$

satisfy

$$\int_{\mathbb{R}^n} f^2(w) d\nu_a(w) - \left(\int_{\mathbb{R}^n} f(w) d\nu_a(w) \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f(w)|^2 d\nu_a(w)$$

and

$$\int_{\mathbb{R}^n} f^2(w) d\nu_b(w) - \left(\int_{\mathbb{R}^n} f(w) d\nu_b(w) \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f(w)|^2 d\nu_b(w)$$

respectively. Now sum the two inequalities above and divide by two to obtain

$$\int_{\mathbb{R}^n} f^2(w) d\nu(w) - \frac{\left(\int_{\mathbb{R}^n} f(w) d\nu_a(w) \right)^2 + \left(\int_{\mathbb{R}^n} f(w) d\nu_b(w) \right)^2}{2} \leq \int_{\mathbb{R}^n} |\nabla f(w)|^2 d\nu(w) \quad (4.2)$$

where $\nu = \frac{\nu_a + \nu_b}{2}$. Moreover

$$\begin{aligned} \left(\int_{\mathbb{R}^n} f(w) d\nu_a(w) \right)^2 &= \left(\int_{\mathbb{R}^n} f(w) p(w - a) dw \right)^2 \\ &= \left(\int_{\mathbb{R}^n} f(w) e^{\langle w, a \rangle - \frac{|a|^2}{2}} p(w) dw \right)^2 \\ &= \int_{\mathbb{R}^n} (f \diamond f)(w) e^{\langle w, a \rangle - \frac{|a|^2}{2}} p(w) dw \\ &= \int_{\mathbb{R}^n} (f \diamond f)(w) d\nu_a(w). \end{aligned}$$

Here we used the characterizing property of the Wick product (2.3) and the fact that the function $e^{\langle w, a \rangle - \frac{|a|^2}{2}}$ plays the role of the stochastic exponential in this framework. If we do the same for the term $\left(\int_{\mathbb{R}^n} f(w) d\nu_b(w) \right)^2$, we can rewrite inequality (4.2) as

$$\int_{\mathbb{R}^n} f^2(w) d\nu(w) - \int_{\mathbb{R}^n} (f \diamond f)(w) d\nu(w) \leq \int_{\mathbb{R}^n} |\nabla f(w)|^2 d\nu(w),$$

which is a particular case of (4.1).

PROOF. First of all we observe that since $F \in \mathbb{D}^{1,3} \subset L^2(W, \mu)$ the Wick product $F \diamond F$ appearing in (4.1) belongs to $\mathcal{G}_{\frac{1}{\sqrt{2}}}$ (see (2.2)) and hence the dual pairing $\langle \langle F \diamond F, \nu \rangle \rangle$ is well defined (according to the assumptions on ν). We divide the proof in two parts.

SECOND INEQUALITY

Define the map

$$\begin{aligned} T : \mathcal{E} &\rightarrow \mathcal{E} \\ F &\mapsto T(F) := F \diamond F - F^2 + \|DF\|_H^2. \end{aligned} \quad (4.3)$$

Since $F \in \mathcal{E}$ we can write $F = \sum_{k=1}^n \lambda_k \mathcal{E}(h_k)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $h_1, \dots, h_n \in H$. Now substitute this expression into (4.3) to obtain

$$\begin{aligned} T(F) &= \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) - \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \cdot \mathcal{E}(h_k) \\ &\quad + \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \cdot \mathcal{E}(h_k) \langle h_j, h_k \rangle_H \\ &= \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) - \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) e^{\langle h_j, h_k \rangle_H} \\ &\quad + \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H \\ &= \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) \left(1 - e^{\langle h_j, h_k \rangle_H} + e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H \right). \end{aligned}$$

Take the expectation with respect to the measure $\nu d\mu$ of the first and last terms of the previous chain of equalities to obtain

$$\begin{aligned} E_\nu[T(F)] &= \sum_{j,k=1}^n \lambda_j \lambda_k E_\nu[\mathcal{E}(h_j) \diamond \mathcal{E}(h_k)] \left(1 - e^{\langle h_j, h_k \rangle_H} + e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H \right) \\ &= \sum_{j,k=1}^n \lambda_j \lambda_k a_{jk} b_{jk}, \end{aligned} \quad (4.4)$$

where for $j, k \in \{1, \dots, n\}$ we set

$$a_{jk} := E_\nu[\mathcal{E}(h_j) \diamond \mathcal{E}(h_k)]$$

and

$$b_{jk} := 1 - e^{\langle h_j, h_k \rangle_H} + e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H.$$

Observe that the matrix $B = \{b_{jk}\}_{1 \leq j, k \leq n}$ is positive semi-definite; in fact, if in the classical Poincaré inequality

$$E[F^2] - E[F]^2 \leq E[\|DF\|_H^2],$$

we take F to be $\sum_{k=1}^n \lambda_k \mathcal{E}(h_k)$ one gets

$$\sum_{j,k=1}^n \lambda_j \lambda_k (e^{\langle h_j, h_k \rangle_H} - 1) \leq \sum_{j,k=1}^n \lambda_j \lambda_k e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H,$$

which corresponds exactly to what we are claiming. On the other hand, the matrix $A = \{a_{jk}\}_{1 \leq j, k \leq n}$ is positive semi-definite since we are assuming ν to be strongly positive (see (3.1)).

Therefore the matrix $A \circ B := \{a_{jk} \cdot b_{jk}\}_{1 \leq j, k \leq n}$ (which corresponds to the Hadamard product of the matrix A with the matrix B) is also positive semi-definite (see for instance Styan [20]), that means

$$\sum_{j,k=1}^n \lambda_j \lambda_k a_{jk} b_{jk} \geq 0,$$

for any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. From (4.4) this corresponds to

$$E_\nu[T(F)] \geq 0 \text{ for all } F \in \mathcal{E}.$$

From the definition of T this is equivalent to

$$E_\nu[F \diamond F] - E_\nu[F^2] + E_\nu[\|DF\|_H^2] \geq 0. \quad (4.5)$$

Therefore we have proved the second inequality in (4.1) for $F \in \mathcal{E}$ (recall that $\langle \langle F \diamond F, \nu \rangle \rangle = E_\nu[F \diamond F]$ since $F \diamond F \in \mathcal{L}^2(W, \mu)$ for $F \in \mathcal{E}$).

The next step is to extend the validity of (4.5) to the whole $\mathbb{D}^{1,3}$. Since the expectations in (4.5) are taken with respect to the measure $\nu d\mu$, we need to control the norms in $\mathcal{L}^2(W, \nu d\mu)$ with the norms in $\mathcal{L}^3(W, \mu)$ (to exploit the density of \mathcal{E} in $\mathbb{D}^{1,3}$). By the Nelson's hyper-contractivity theorem and the assumptions on ν we have that

$$\begin{aligned} E[|\nu|^3]^{\frac{1}{3}} &= E[|\Gamma(1/\sqrt{2})\Gamma(\sqrt{2})\nu|^3]^{\frac{1}{3}} \\ &\leq E[|\Gamma(\sqrt{2})\nu|^2]^{\frac{1}{2}} \\ &< +\infty \end{aligned}$$

showing that $\nu \in \mathcal{L}^3(W, \mu)$. Hence by the Hölder inequality we deduce

$$\begin{aligned} E_\nu[|F|^2]^{\frac{1}{2}} &= E[|F|^2 \cdot \nu]^{\frac{1}{2}} \\ &\leq (E[|F|^3]^{\frac{2}{3}} \cdot E[|\nu|^3]^{\frac{1}{3}})^{\frac{1}{2}} \\ &= CE[|F|^3]^{\frac{1}{3}} \end{aligned}$$

This shows that if $F_n \in \mathcal{E}$ converges to F in $\mathbb{D}^{1,3}$ then $E_\nu[F_n^2]$ and $E_\nu[\|DF_n\|_H^2]$ converge respectively to $E_\nu[F^2]$ and $E_\nu[\|DF\|_H^2]$. Moreover since $E_\nu[F_n \diamond F_n] = \langle \langle F_n \diamond F_n, \nu \rangle \rangle$ and convergence in $\mathbb{D}^{1,3}$ implies convergence in \mathcal{G}^* (where the Wick product is continuous) we can conclude that $E_\nu[F_n \diamond F_n]$ converges to $\langle \langle F \diamond F, \nu \rangle \rangle$. We can therefore extend the validity of (4.5) to the whole $\mathbb{D}^{1,3}$.

FIRST INEQUALITY

Let F be of the form $p(\delta(h_1), \dots, \delta(h_n))$ where $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial and $h_1, \dots, h_n \in H$. Functions of this type belong to \mathcal{G} and they are dense in $\mathcal{L}^p(W, \mu)$ and $\mathbb{D}^{k,p}$ for any $p \geq 1$ and $k \in \mathbb{N}$. Then we can write

$$F \cdot F = F \diamond F + \sum_{j \geq 1} \frac{1}{j!} \sum_{k \geq 1} \langle D^j F, e_k^{(j)} \rangle_{H^{\otimes j}} \diamond \langle D^j F, e_k^{(j)} \rangle_{H^{\otimes j}}, \quad (4.6)$$

where for any $j \geq 1$, $\{e_k^{(j)}\}_{k \geq 1}$ is an orthonormal basis of $H^{\otimes j}$.

Formula (4.6) is the dual of (1.4); it also appeared for the first time in [17] and independently in [11] without proof; see [12] and [14] for the proof.

Observe that the sum over j is finite since p is a polynomial; moreover rewriting $p(\delta(h_1), \dots, \delta(h_n))$ as $\tilde{p}(\delta(\tilde{h}_1), \dots, \delta(\tilde{h}_k))$ with $k \leq n$, \tilde{p} a polynomial and $\{\tilde{h}_1, \dots, \tilde{h}_k\}$ an orthonormal basis for $\text{span}\{h_1, \dots, h_n\}$ one can make also the sum over k to be finite (choosing proper basis $\{e_k^{(j)}\}_{k \geq 1}$). Hence

$$\begin{aligned} E_\nu[F^2] &= E_\nu[F \diamond F] + \sum_{j \geq 1} \frac{1}{j!} \sum_{k \geq 1} E_\nu[(\langle D^j F, e_k^{(j)} \rangle_{H^{\otimes j}})^{\diamond 2}] \\ &\geq E_\nu[F \diamond F], \end{aligned}$$

since all the terms inside the last sum are non negative (due to the strong positivity of ν (see 3.1)). This shows that

$$E_\nu[F^2] - E_\nu[F \diamond F] \geq 0$$

or equivalently

$$E_\nu[F^2] - \langle \langle F \diamond F, \nu \rangle \rangle \geq 0.$$

for any F of polynomial type. The density of these functions in $\mathbb{D}^{1,3}$, the continuity of the Wick product in \mathcal{G}^* and the assumptions on ν complete the proof. \blacksquare

Remark 4.2 *Theorem 4.1 is a generalization of the classical Poincaré inequality on an abstract Wiener space, i.e.*

$$E[F^2] - E[F]^2 \leq E[\|DF\|_H^2], \quad F \in \mathbb{D}^{1,2}. \quad (4.7)$$

In fact, choosing $\nu = 1$ in (4.1), which clearly is strongly positive and belongs to $\mathcal{G}_{\sqrt{2}}$, one gets precisely (4.7) (recall equality (2.4) and the fact for $\nu = 1$ we can replace $\mathbb{D}^{1,3}$ with $\mathbb{D}^{1,2}$). Moreover, the family of probability measures for which our Poincaré inequality is proved, namely measures of the type $\nu d\mu$ with ν strongly positive, is in general not included in the class of log-concave measures, which is the class of probability measure considered by Brascamp and Lieb [2] (see the example at the end of Section 3) and Feyel and Üstünel [6] (in the abstract Wiener space setting).

Remark 4.3 If $F \in \mathbb{D}^{1,3}$ and ν is a strongly positive element of $\mathcal{G}_{\sqrt{2}}$ then we have

$$(E_\nu[F])^2 \leq \langle \langle F \diamond F, \nu \rangle \rangle \leq E_\nu[F^2].$$

The second inequality is part of Theorem 4.1. The first inequality is easily obtained as follows:

$$\begin{aligned} 0 &\leq E_\nu[(\varphi - E_\nu[\varphi]) \diamond (\varphi - E_\nu[\varphi])] \\ &= E_\nu[\varphi \diamond \varphi] - (E_\nu[\varphi])^2. \end{aligned}$$

Here $\varphi \in \mathcal{E}$ and we utilized the property that $\varphi \diamond \psi = \varphi \cdot \psi$ if φ or ψ is constant. Therefore

$$\begin{aligned} E_\nu[F^2] - \langle \langle F \diamond F, \nu \rangle \rangle &\leq E_\nu[F^2] - (E_\nu[F])^2 \\ &= \text{Var}_\nu(F) \end{aligned}$$

where $\text{Var}_\nu(F)$ stands for the variance of F under the measure $\nu d\mu$.

Remark 4.4 Theorem 4.1 has been proved for functions in $\mathbb{D}^{1,3}$; the natural space would be $\mathbb{D}_\nu^{1,2}$ where we mean the Sobolev space with respect to the measure $\nu d\mu$. This space is well defined through the closability of the gradient operator with respect to the norm in $\mathcal{L}^2(W, \nu d\mu)$ which is however not known in the general case. This is why for instance the Poincaré inequality for log-concave measures on abstract Wiener spaces obtained in [6] is proved only for cylindrical smooth functions.

4.1 A refinement on the classical Wiener space

In this section we are going to generalize Theorem 4.1 in the spirit of the Poincaré type inequality obtained by Houdré and Kagan in [8] for the finite dimensional case and later on by Houdré and Pérez-Abreu in [9] for the classical Wiener space. More precisely, in the paper [9] it is proved that on the classical Wiener space the following inequality holds for any $F \in \mathbb{D}^{2k,2}$ with $k \in \mathbb{N}$:

$$\sum_{l=1}^{2k} \frac{(-1)^{l+1}}{l!} E[\|D^l F\|_{H^{\otimes l}}^2] \leq E[|F|^2] - E[F]^2 \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} E[\|D^l F\|_{H^{\otimes l}}^2]. \quad (4.8)$$

Its proof relies on iterations of the Clark-Ocone formula (see [16]) and we do not know whether inequality (4.8) is valid in the abstract Wiener space setting; since the proof of our next theorem will rely on the validity of inequality (4.8) we are forced to work in the framework of the classical Wiener space.

We will generalize the second inequality in (4.8) replacing the expectations $E[\cdot]$ with $E_\nu[\cdot]$, where ν is a strongly positive element of $\mathcal{G}_{\sqrt{2}}$, and the term $E[F]^2$ with $\langle \langle F \diamond F, \nu \rangle \rangle$. As before, these two last expressions coincide for $\nu = 1$.

Theorem 4.5 Let (H, W, μ) be the classical Wiener space and let $\nu \in \mathcal{G}_{\sqrt{2}}$ be strongly positive. Then for any $k \in \mathbb{N}$ and $F \in \mathbb{D}^{2k-1,3}$ we have

$$0 \leq E_\nu[|F|^2] - \langle \langle F \diamond F, \nu \rangle \rangle \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} E_\nu[\|D^l F\|_{H^{\otimes l}}^2]. \quad (4.9)$$

PROOF. The proof is similar to one of Theorem 4.1; one has to start with the map

$$\begin{aligned} T : \mathcal{E} &\rightarrow \mathcal{E} \\ F &\mapsto T(F) := F \diamond F - F^2 + \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} \|D^l F\|_{H^{\otimes l}}^2 \end{aligned} \quad (4.10)$$

and observe that for $F = \sum_{k=1}^n \lambda_k \mathcal{E}(h_k)$ one gets

$$T(F) = \sum_{j,k=1}^n \lambda_j \lambda_k \mathcal{E}(h_j) \diamond \mathcal{E}(h_k) \left(1 - e^{\langle h_j, h_k \rangle_H} + \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H^l \right).$$

Now the matrix $A = \{a_{i,j}\}_{1 \leq i,j \leq n}$ defined by

$$a_{ij} := 1 - e^{\langle h_j, h_k \rangle_H} + \sum_{l=1}^{2k-1} \frac{(-1)^{l+1}}{l!} e^{\langle h_j, h_k \rangle_H} \langle h_j, h_k \rangle_H^l$$

is positive semi-definite by the second inequality in (4.8) applied to $F = \sum_{k=1}^n \lambda_k \mathcal{E}(h_k)$. The proof then follows as before. \blacksquare

References

- [1] L. Bertini and N. Cancrini, The stochastic heat equation: Feynman-Kac formula and intermittence, *J. Stat. Phys.* **78** (1995) 1377-1401.
- [2] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* **22** (1976) 366-389.
- [3] H. Chernoff, A note on an inequality involving the normal distribution, *Ann. Probab.* **9** (1981) 533-535.
- [4] P. Da Pelo, A. Lanconelli and A. I. Stan, A Hölder-Young-Lieb inequality for norms of Gaussian Wick products, *Inf. Dim. Anal. Quantum Prob. Related Topics* **14** (2011) 375-407.
- [5] P. Da Pelo, A. Lanconelli and A. I. Stan, An Itô formula for a family of stochastic integrals and related Wong-Zakai theorems, *Stochastic Processes and their Applications* **123** (2013) 3183-3200.
- [6] D. Feyel and A. S. Üstünel, The notion of convexity and concavity on Wiener space, *J. Funct. Anal.* **176** (2000) 400-428.
- [7] L. Gross, Logarithmic Sobolev inequality, *Amer. J. Math.* **97** (1975) 1061-1083.
- [8] C. Houdré and A. Kagan, Variance inequalities for functions of Gaussian variable, *J. Theoret. Probab.* **8** (1995) 23-30.
- [9] C. Houdré and V. Pérez-Abreu, Covariance identities and inequalities for functionals on Wiener and Poisson spaces, *Ann. Probab.* **23** (1995) 400-419.

- [10] Y. Hu, Chaos expansion of heat equations with white noise potentials, *Potential Analysis* **16** (2002) 45-66.
- [11] Y. Hu and B. Øksendal, Wick approximation of quasilinear stochastic differential equations, *Stochastic analysis and related topics V, Progr. Probab.* **38** (1996) 203–231.
- [12] Y. Hu and J. Yan, Wick calculus for nonlinear Gaussian functionals, *Acta Math. Appl. Sinica* **25** (2009) 399-414.
- [13] S. Janson, *Gaussian Hilbert spaces*, Cambridge Tracts in Mathematics, 129. Cambridge University Press, Cambridge, 1997.
- [14] A. Lanconelli, On the extension of a basic property of conditional expectations to second quantization operators, *Comm. Stoch. Anal.* **3** (2009) 369-381.
- [15] J. Nash, Continuity of solutions of partial and elliptic equations, *Amer. J. Math.* **80** 931-954.
- [16] D. Nualart, *Malliavin calculus and Related Topics, II edition*, Springer, New York 2006
- [17] D. Nualart and M. Zakai, Positive and strongly positive Wiener functionals, *Barcelona Seminar on Stochastic Analysis Progr. Probab.* **32** (1991) 132–146.
- [18] J. Potthoff and M. Timpel, On a dual pair of spaces of smooth and generalized random variables, *Potential Analysis* **4** (1995) 637–654.
- [19] D. W. Stroock, Homogenous chaos revisited, *Seminaire de Probabilités XXI*, Lecture Notes in Math. **1247** (1987) 1-8.
- [20] G. Styan, Hadamard product and multivariate statistical analysis, *Linear algebra and its applications* **6** (1973) 217-240.